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THE PERFORMANCE OF NONPARAMETRIC DETECTORS USING SPECTRAL ESTIM--ETC(U)
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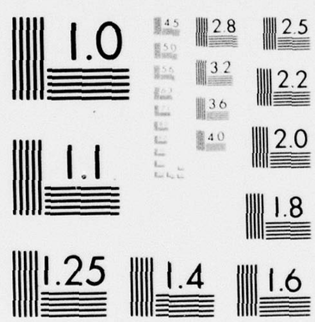

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THE PERFORMANCE OF NONPARAMETRIC DETECTORS USING SPECTRAL ESTIMATES

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ABSTRACT

M.N. Woinsky proposed the application of various non-parametric statistics to spectral data in order to detect narrow band signals. He determined the asymptotic relative efficiency (ARE) of two such tests. Hansen and Olsen have found the distribution of ranks of spectral data for two types of signal embedded in noise. Here the distribution of ranks for a third type of signal is derived. The performance of various non-parametric statistics for these three signals are calculated.

I. INTRODUCTION

Following Woinsky [1] we consider the situation in which a signal embedded in a time series appears as a narrow line in the frequency domain. Detection is based on the matrix of spectral estimates X_{ij}: $i=1,2,\ldots,M$; $j=1,2,\ldots,N$. Here time is referenced by the i parameter and frequency by the j parameter. The parameter N is the largest number of frequency estimates for which the spectrum can be said to be flat, while M is the number of spectral estimates per frequency cell to be used in the decision process.

Let R_{ij} be the rank of the X_{ij} for a fixed time interval, that is i fixed, $j=1,2,\ldots,N$. The null hypothesis is that all N frequencies contain noise only. For the X_{ij} independent identically distributed random variables, the R_{ij} are uniformly distributed over the integers $1,2,\ldots N$. The alternative hypothesis is that a signal occurs at one of the frequencies represented. In order to determine the distribution of the ranks in this case, one must assume a specific distribution for the X_{ij}; that is, one must specify the nature of the signal and the noise.

II. DISTRIBUTION OF THE RANK STATISTICS

Suppose $X_1, X_2, \ldots, X_N$ represent a set of N independent random variable spectral estimates of which the first N-1 correspond to frequencies without signal components and of which the last, X_N, corresponds to a frequency cell at which a signal occurs. X_N is denoted as the signal random variable. We consider the case where the first N-1 estimates are exponential random variables with mean Ψ; that is, the time-bandwidth product for the assumed spectral analyzer is unity and Ψ represents the expected power in each cell. This is a reasonable assumption as theoretical [1], [2] and empirical studies [3] have shown noise spectral estimates to be independent exponential random variables for a variety of time series.

Three types of signal are considered below. Case I refers to an X_N which is an exponential random variable with parameter $\Theta(1+\Psi)$; that is, the signal is a "Gaussian target" and Θ is cell signal-to-noise power ratio [4].

Case II refers to a distribution of X_N defined by

$$f(x_N) = \frac{1}{2}\Psi^{-1} \exp(-\frac{1}{2}\Psi^{-1} x_N+\Theta)\ I_o\left[(2\Psi^{-1}\theta x_N)^{\frac{1}{2}}\right] \ldots.(1)$$

where $x_N \geq 0$ and $I_o(x)$ is the modified Bessel function of order zero. This distribution occurs when the signal acts like a sinusoidal target [5] [6].

Case III, formulated by Swerling for a model of a fluctuating radar target [6] [7], refers to a density function of the form

$$f(x_N) = \frac{1}{2} b^2\Psi^{-1}(1 + ax_N\Psi^{-1}) \exp(-\frac{1}{2} x_N b\Psi^{-1}) \ldots.. \quad (2)$$

where $\quad b$ represents $(1 + \frac{1}{2}\theta)^{-1}$

and $\quad a$ represents $\left[2 + (4/\theta)\right]^{-1}$.

Let P(k) be the probability that the signal frequency takes rank k, where k = 1,2,...N. This occurs if there are k-1 noise random variables less than X_N and N-k such random variables greater than X_N.

We find the probability of one such ordering and then multiply by $\binom{N-1}{k-1}$ for the final expression for P(k). Now since $X_1,\ldots,X_N$ are independent, the probability of one specific ordering is

$$\int f(x)\ P\begin{bmatrix}\text{k-1 specified noise fre-}\\ \text{quency spectral estimates}\\ \text{are less than x}\end{bmatrix}\ P\begin{bmatrix}\text{remaining N-k}\\ \text{estimates are}\\ \text{greater than x}\end{bmatrix} dx \ldots(3)$$

where P denotes the probability of an event and f(x) is the density function of X_N.

(a) CASE I

In Case I, expression (3) takes the form

$$\int_0^\infty \frac{1}{2\Psi(1+\Theta)} \exp\left[-{}^{x}/_{2\Psi(1+\Theta)}\right] \left[\frac{1}{2\psi}\int_0^x \exp(-{}^{y}/_{2\Psi})dy\right]^{k-1} \left[\frac{1}{2\psi}\int_x^\infty \exp(-{}^{y}/_{2\Psi})dy\right]^{N-k} dx$$

$$= \frac{1}{2\Psi(1+\Theta)} \int_0^\infty \left[1-\exp(-{}^{x}/_{2\Psi})\right]^{k-1} \left[\exp(-{}^{x}/_{2\psi})\right]^{N-k} \exp(-{}^{x}/_{2\Psi(1+\Theta)})dx$$

$$= \alpha \int_0^\infty [1-\exp(-y)]^{k-1} [\exp(-y)]^{N-k} \exp(-\alpha y)dy,$$

where α represents $(1+\theta)^{-1}$ and y represents $x/(2\Psi)$.

Now, by [8] page 305 Formula 3.312-1,

$$\int_0^\infty \left[1-\exp(-{}^{x}/_{\beta})\right]^{\nu-1} \exp(-\mu x)dx = \beta B(\beta\mu,\nu)$$

if Re $\beta>o$, Re $\nu>o$, Re $\mu>o$, where $B(x,y)$ represents the Beta function.

Thus our integral reduces to

$$\alpha B(N-k+\alpha,k) = \alpha\Gamma(N-k+\alpha)\Gamma(k)/\Gamma(N+\alpha)$$

We note that the conditions on β and ν are always satisfied; care must be taken with the condition on μ which reduces to the requirement that $N+\alpha>k$. This holds if $\alpha>0$. If $\alpha=0$ we have the case where cell signal-to-noise power ratio is infinite. It can be independently shown that the same result holds in this case.

Finally, multiplication by $\binom{N-1}{k-1}$ gives the result

$$P(k) = P(k,N,\alpha) = \frac{\alpha\Gamma(N)\Gamma(N+\alpha-k)}{\Gamma(N+\alpha)\Gamma(N-k+1)} \quad \ldots(4)$$

for k = 1,...N. This equation is equivalent, although simpler in form, to equation (32) of [9].

It is easily seen that if Θ is zero, P(k) is N^{-1} for all k, the uniform distribution. The ranks are uniformly distributed for any

distribution of the X_i as long as the X_i are independent and identically distributed.

It can be shown that $P(k) \leq P(k')$ for $k < k'$, with equality holding only if θ equals zero.

(b) CASE II

In this case expression (3) becomes the integral

$$\int_o^\infty \left[1-\exp(-y)\right]^{k-1} \left[\exp(-y)\right]^{N-k} \exp\left[-(y+\theta)\right] I_o\left[2(y\theta)^{\frac{1}{2}}\right] dy$$

$$= \exp(-\Theta) \int_o^\infty \left[1-\exp(-y)\right]^{k-1} \exp\left[-y(N-k+1)\right] I_o\left[2(y\Theta)^{\frac{1}{2}}\right] dy$$

$$= \exp(-\Theta) \sum_{i=o}^{k-1} (-1)^i \binom{k-1}{i} \int_o^\infty \exp\left[-y(N-k+i+1)\right] I_o\left[2(y\Theta)^{\frac{1}{2}}\right] dy$$

where y represents $\frac{1}{2} x \Psi^{-1}$.

Now, by [10] page 1026 Formulae 29.3.81,

$$\int_o^\infty \exp(-pt) I_o\left[2(at)^{\frac{1}{2}}\right] dt = p^{-1} \exp(a/p), \text{ where Re } p>0.$$

Hence our integral becomes

$$\exp(-\Theta) \sum_{i=o}^{k-1} \binom{k-1}{i} (-1)^i \frac{1}{N-k+i+1} \exp\left[\Theta/(N-k+i+1)\right]$$

whenever Re N-k+i+k>0. This condition is always satisfied.

Multiplying by $\binom{N-1}{k-1}$ yields, after simplifying,

$$P(k) = P(k,N,\Theta) = \frac{\Gamma(N)\exp(-\Theta)}{\Gamma(N-k+1)} \sum_{i=o}^{k=1} \frac{(-1)^i \exp\left[\Theta/(N+1-k+i)\right]}{(N+1-k+i)\Gamma(i+1)\Gamma(k-i)} \quad \ldots .(5)$$

for $k = 1,2,\ldots N$.

This is identical to equation (30) of [9].

Equation (5) gives the density function for Case II. Care must be taken in its evaluation, as we are summing a strictly alternating series whose sum while lying between zero and one contains terms of different orders of magnitude. It can be shown analytically that

$$P(k) = \frac{1}{N} \text{ if } \theta = 0.$$

(c) CASE III

For the density function given by Equation (2), expression (3) becomes

$$\frac{1}{2}\int_o^\infty b^2\Psi^{-1}(1+ax\Psi^{-1})\exp(-\tfrac{1}{2}xb\Psi^{-1})\left[\exp(-\tfrac{1}{2}x\Psi^{-1})\right]^{N-k}\left[1-\exp(-\tfrac{1}{2}x\Psi^{-1})\right]^{k-1}dx$$

$$= b^2\left\{\int_o^\infty \left[1-\exp(-y)\right]^{k-1}\exp\left[-y(N-k+b)\right]dy + 2a\int_o^\infty\left[1-\exp(-y)\right]^{k-1}\exp\left[-y(N-k+b)\right]ydy\right\}$$

where y represents $\frac{1}{2}x\Psi^{-1}$.

Now,

$$\int_o^\infty\left[1-\exp(-y)\right]^{k-1}\exp\left[-y(N-k+b)\right]dy$$

$$= B(N+k+b,k)$$

$$= \Gamma(N-k+b)\Gamma(k)/\Gamma(N+b).$$

Also, by [11] formula 5.9,

$$\int_o^\infty\left[1-\exp(-y)\right]^{k-1}\exp\left[-y(N-k+b)\right]ydy$$

$$= B(N-k+b,k)\left[D(N+b) - D(N-k+b)\right]$$

where D represents the Digamma function. From [10] formula 6.36 the expression in the square bracket equals $\sum_{m=o}^{k-1}(m+N-k+b)^{-1}$, multiplication by $\binom{N-1}{k-1}$ yields

$$P(k) = P(k,N,b) = b^2 \, (N-k+b) \left[(N+b)\,(N-k+1)\right]^{-1} \left[1 + 2a \sum_{m=o}^{k-1} (m+N+b-k)^{-1}\right]$$

As in Case I it can be shown analytically that $P(k) \leq P(k')$ for $k < k'$ with equality holding only if θ equals zero.

Figure 1 contains typical density functions when $N = 8$. For low cell signal-to-noise ratios these functions are almost the same (Fig.1(a)), whereas for larger ratios that of Case II tends to the limiting case $P(N) = 1$ "faster" than the density function of Case I. The results for Case III are intermediate, and are not included in the Figures.

III. DETECTOR PERFORMANCE

The modified nonparametric decision scheme is based on the sum of M independent random variables representing some function of the rank at each frequency. That is, we decide a signal is present at frequency j depending on the value of

$$S_j = \frac{1}{M} \sum_{i=1}^{M} a(R_{ij}) \ldots \tag{7}$$

where $a(x)$ is some function of x. For M small we use numerical convolution, for M large we apply the Central Limit Theorem to obtain the density function of S_j.

We determine S_j when:

(1) $a(R_{ij}) = R_{ij}$ (modified Mann-Whitney statistic).

(2) $a(R_{ij}) = \sum_{\ell=N-R_{ij}+1}^{N} \ell^{-1}$ (modified Savage test).

(3) $a(R_{ij}) = \text{sign}(R_{ij} - N/2)$ (modified Median test).

Dillard [6] has studied the case $a(R_{ij}) = \text{sign}\,(R_{ij}-N)$. The results above would enable us to study the detection $a(R_{ij}) = \text{sign}\,(R_{ij}-kN)$ for $0 < k \leq 1$. In these cases of course tables of the binomial distribution may be used to determine false alarm and detection probabilities.

IV RESULTS

Computer programs were written to determine the performance under various combinations of N, M and Θ of the three detectors above in Cases I, II and III.

Figure 2 gives some typical results. In most cases we present the Savage statistic results, the "best" detector considered here.

In all cases considered, the Savage detector gave better detection results than the Mann-Whitney detector. In turn the Mann-Whitney performed better than the Median detector. Woinsky [1], [12] has shown with regard to the ARE that the Savage statistic performs better than the Mann-Whitney. Figure 2(a) gives a typical result for a Case II situation with low signal-to-noise ratio and a large sample size. Case I, II and III signals have the same performance for the various detectors at very low signal-to-noise ratios, at other values for a fixed false alarm rate we can detect a sinusoidal target with greatest probability. Figures 2(b) and (c) illustrate this.

V. CONCLUSIONS

We have determined the density functions of the rank statistic for three realistic signal models, hence exact receiver performance can be evaluated for any practical detector based on these ranks.

Illustrative numerical results were presented for three well-known rank statistics.

The result of Woinsky indicating the superiority of the Savage statistic over the Mann-Whitney statistic in the ARE sense has been found applicable to all cases tested through numerical evaluation of the exact expressions derived over a wide range of signal-to-noise ratio and sample size.

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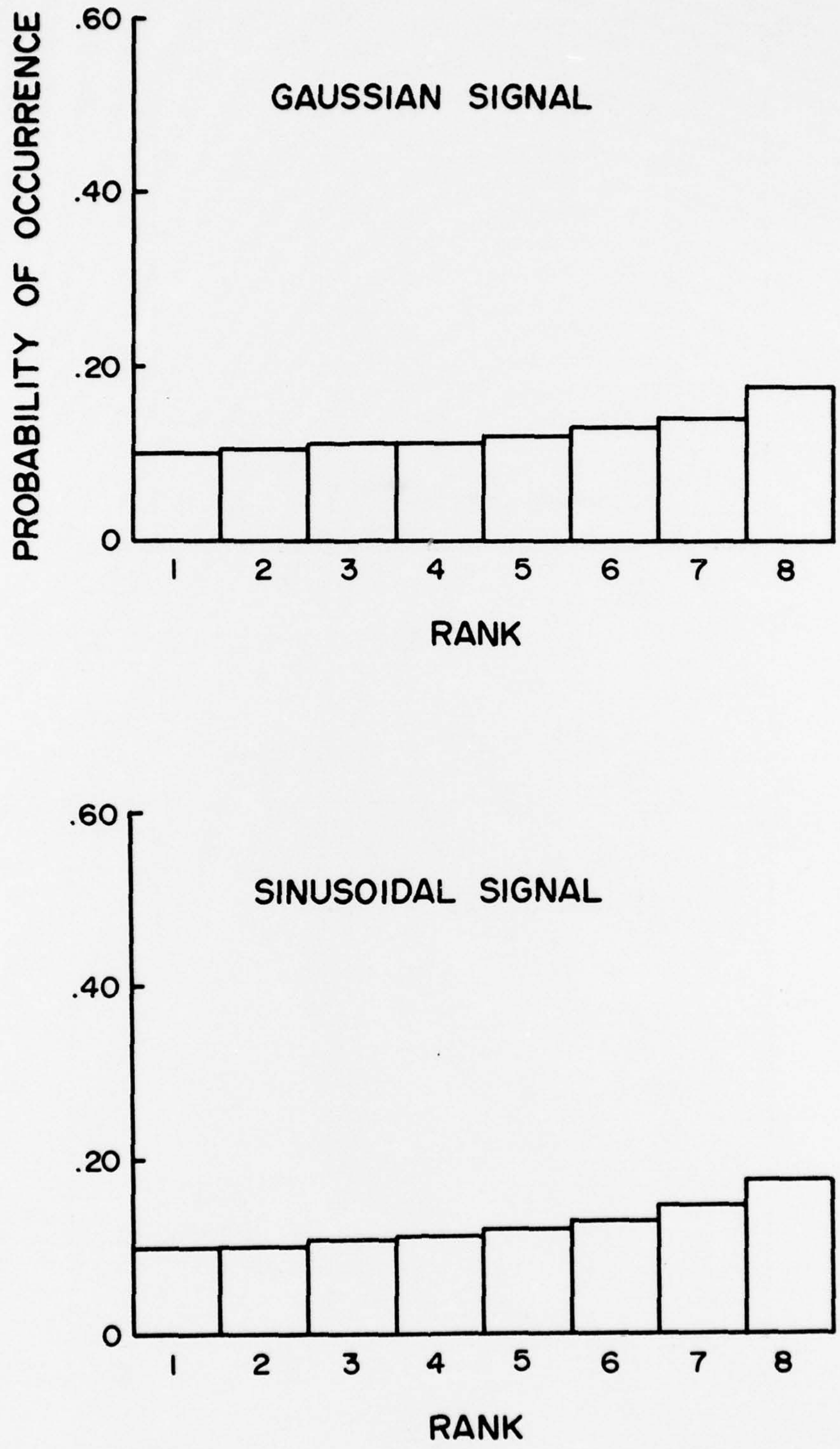

Figure I(a). Rank versus probability of occurence for Gaussian and sinusoidal signals in cell containing signal. Cell signal-to-noise ratio -6 dB, 8 frequencies ranked.

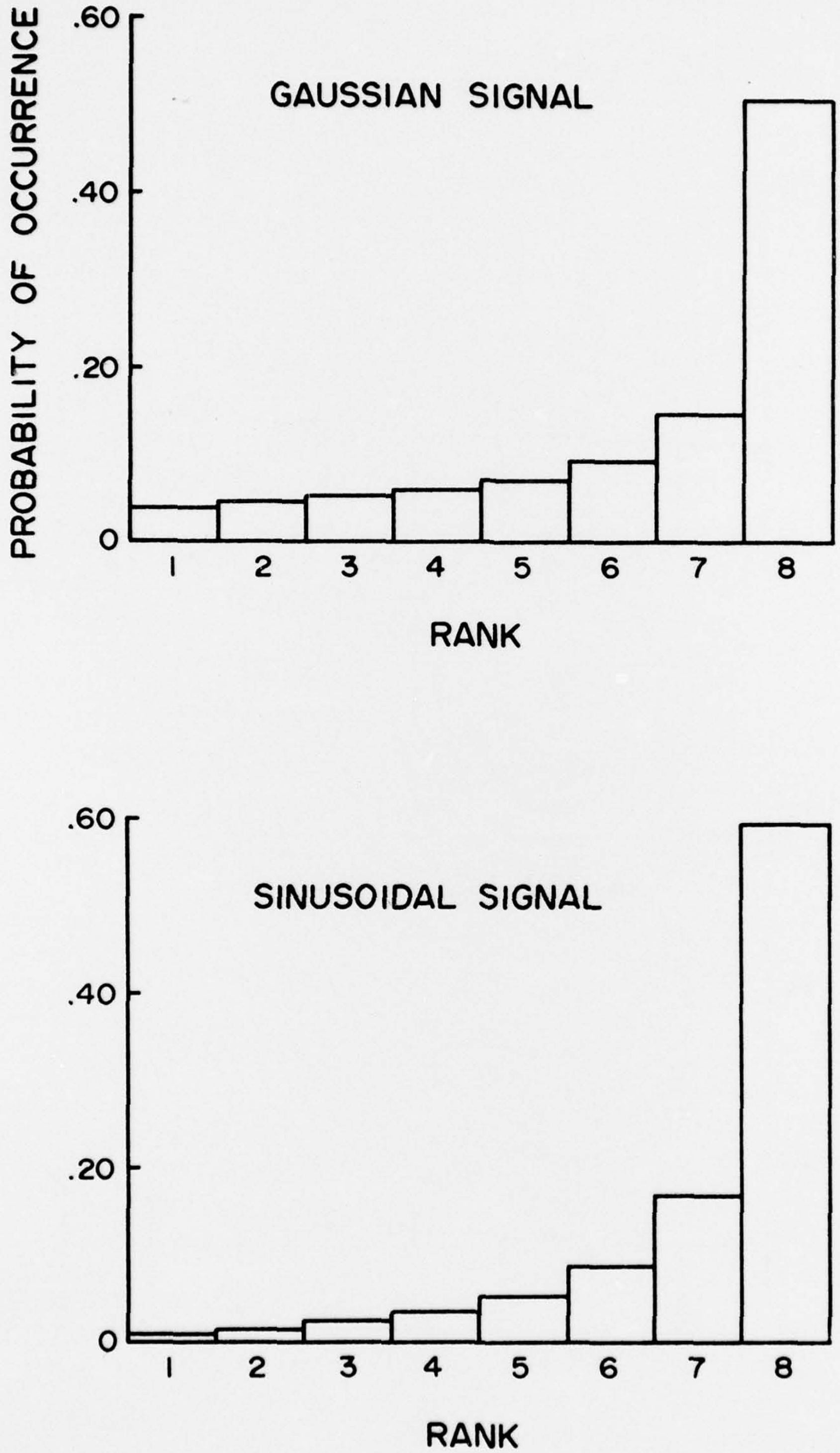

Figure I(b). Rank versus probability of occurrence for Gaussian and sinusoidal signals in cell containing signal. Cell signal-to noise ratio 4 dB, 8 frequencies ranked.

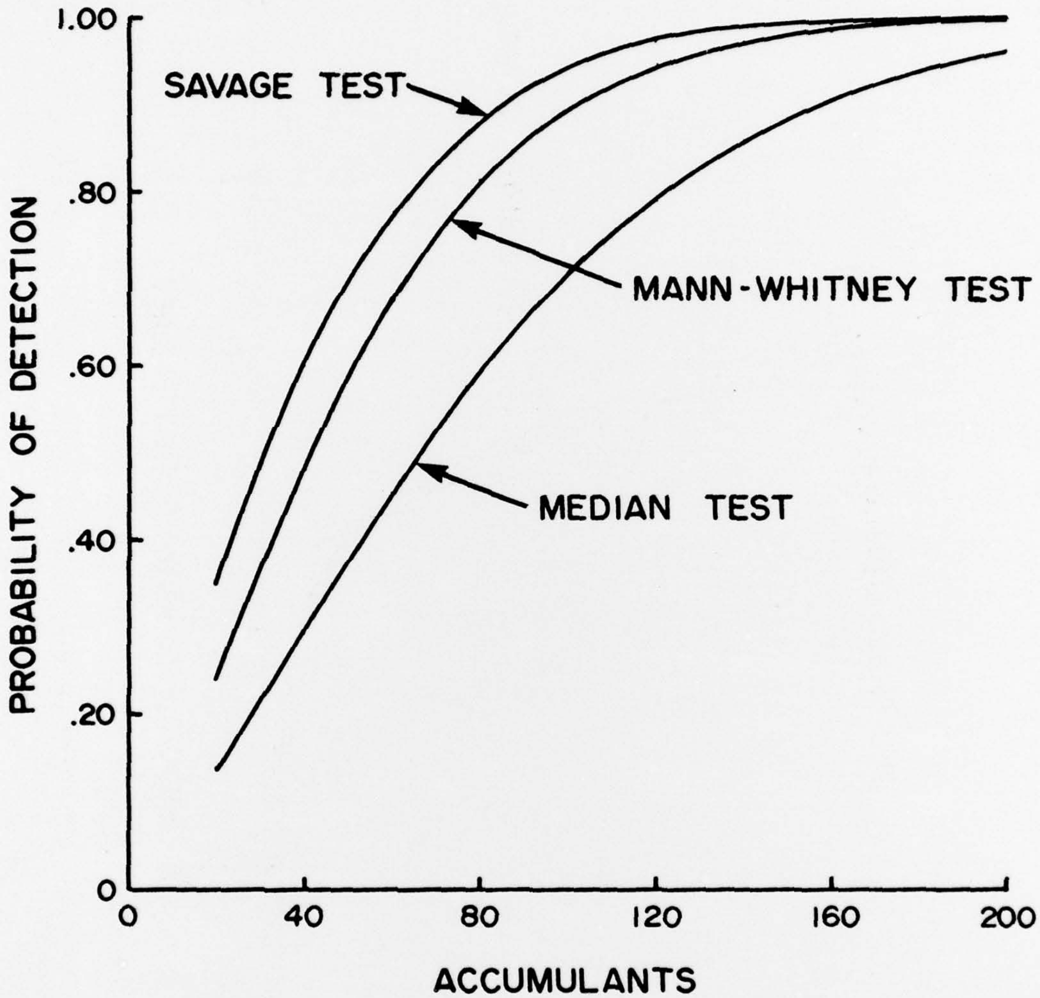

Figure II(a). Accumulants versus probability of detection for three statistical tests. False alarm rate 10^{-2}; cell signal-to-noise ratio -3 dB; 16 frequencies ranked; sinusoidal signal.

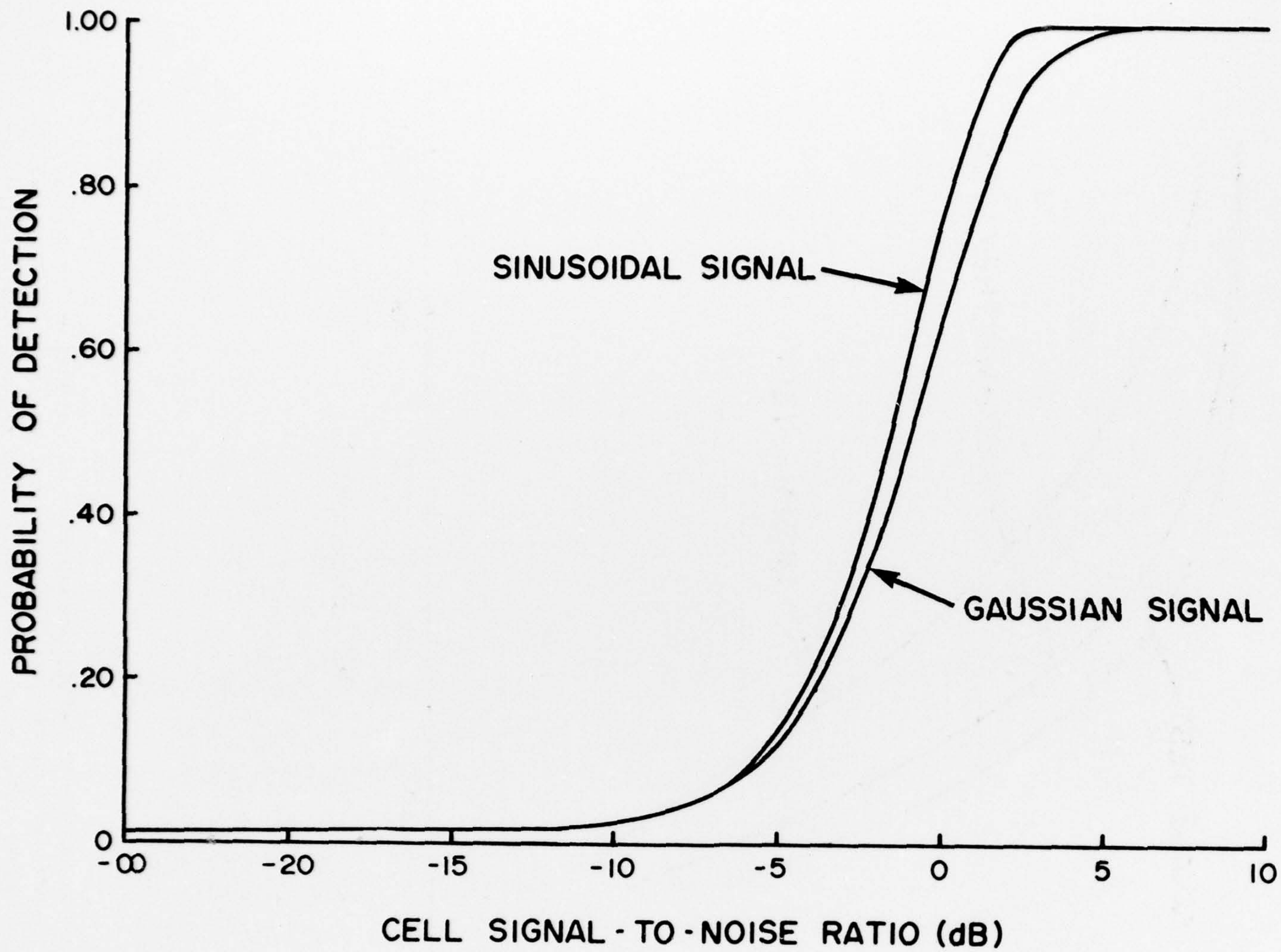

Figure II(b). Cell signal-to-noise ratio versus probability of detection for sinusoidal and Gaussian signals. Savage Test; false alarm rate 10^{-2}; 20 accumulants; 8 frequencies ranked.

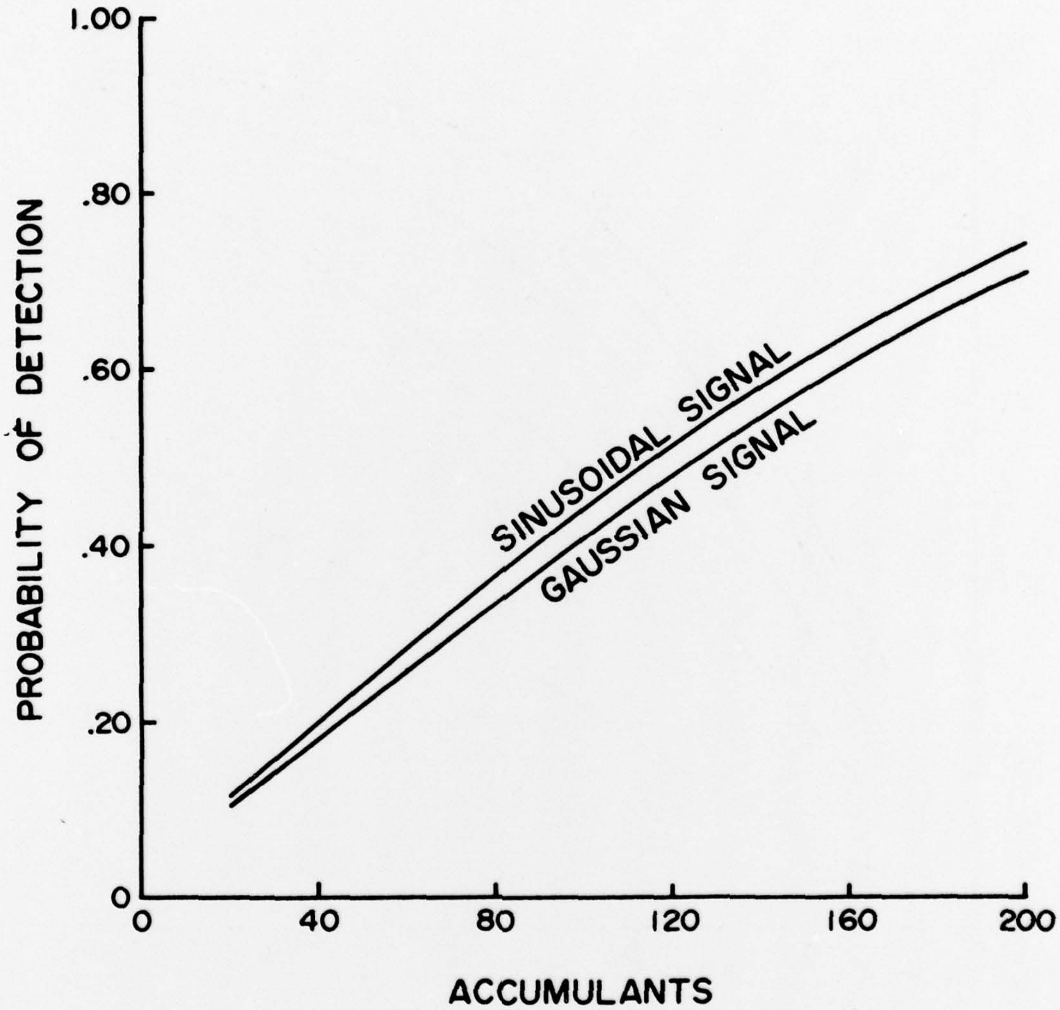

Figure II(c). Accumulants versus probability of detection for sinusoidal and Gaussian signals. Savage Test; 16 frequencies ranked; cell signal-to-noise ratio -6 dB, false alarm rate 10^{-2}.

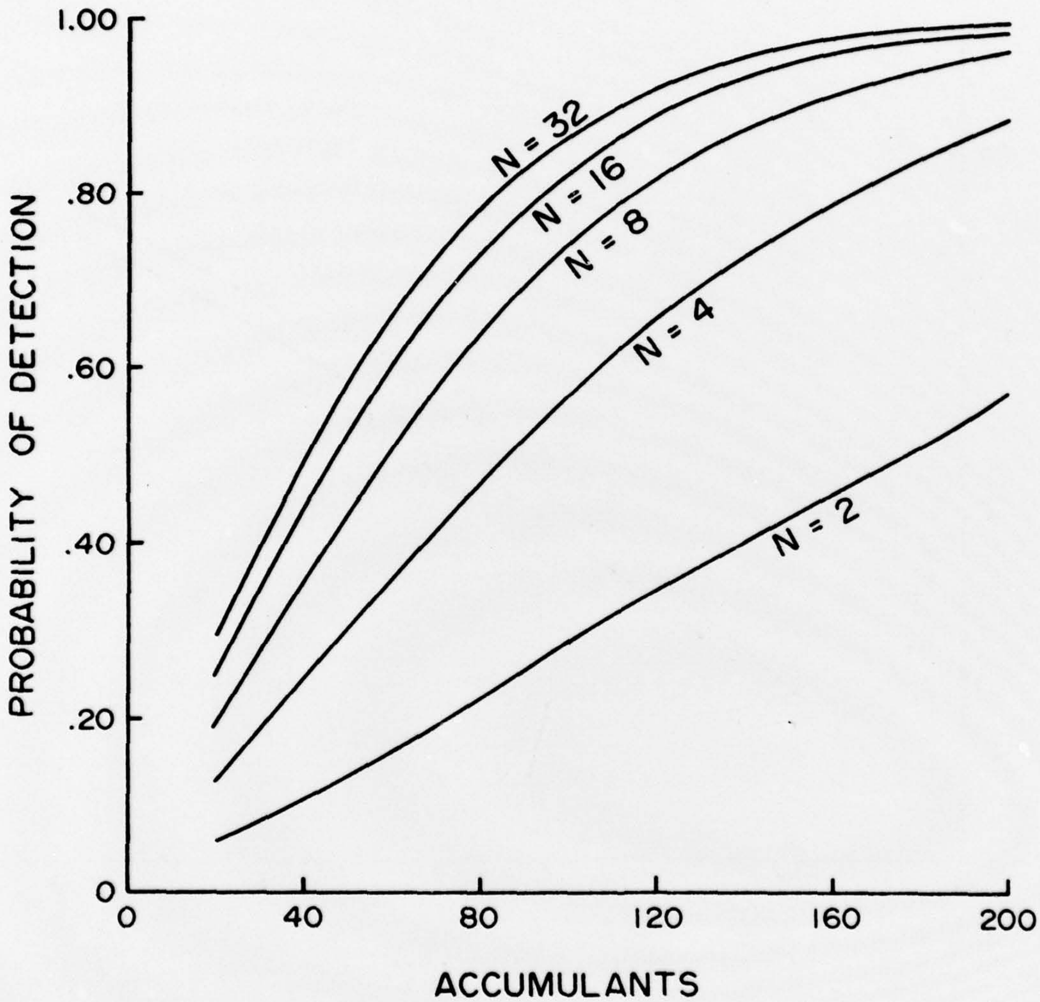

Figure II(d). Accumulants versus probability of detection for various numbers of frequencies ranked. Savage Test, cell signal-to-noise ratio -3.68 dB, false alarm rate 10^{-2}; Gaussian signal.